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IDENTITIES FOR F_{kn} AND L_{kn}

B-115 Proposed by H. H. Ferns, Victoria, B.C., Canada

From the formulas of B-106:

$$2F_{i+j} = F_iL_j + F_jL_i$$
$$2L_{i+j} = 5F_iF_j + L_iL_j$$

one has

$$F_{2n} = F_nL_n$$
$$F_{3n} = (5F_n^3 + 3F_nL_n^2)/4$$
$$L_{2n} = (5F_n^2 + L_n^2)/2$$
$$L_{3n} = (15F_n^2L_n + L_n^3)/4$$

Find and prove the general formulas of these types.

Solution by Stanley Rabinowitz, Far Rockaway, New York.

The formulas look neater when expressed in matrix form. Putting $i = (k-1)n$ and $j = n$ in the formulas of B-106 gives

(R) $$\begin{pmatrix} F_{kn} \\ L_{kn} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} L_n & F_n \\ 5F_n & L_n \end{pmatrix}\begin{pmatrix} F_{(k-1)n} \\ L_{(k-1)n} \end{pmatrix}$$

Repeated application of this formula gives the desired solution:

$$\begin{pmatrix} F_{kn} \\ L_{kn} \end{pmatrix} = \frac{1}{2^k}\begin{pmatrix} L_n & F_n \\ 5F_n & L_n \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

since $F_0 = 0$ and $L_0 = 2$.

Note: From (R) or the formulas of B-106, one can obtain the proposer's formulas:

$$F_{(k+1)n} = \frac{1}{2^k}\sum_{i=0}^{[k/2]} 5^i \binom{k+1}{k-2i} F_n^{2i+1} L_n^{k-2i} ,$$

$$L_{(k+1)n} = \frac{1}{2^k}\sum_{i=0}^{[(k+1)/2]} 5^i \binom{k+1}{k+1-2i} F_n^{2i} L_n^{k+1-2i} .$$